5.3.4 変動成分に関する方程式

乱れの運動エネルギー $\overline{u_i'u_i'}/2$ を支配する式

(5.33) に u_i' をかけて平均をとると

$$\overline{u_i'\frac{\partial(\overline{u_i}+u_i')}{\partial t}+u_i'(\overline{u_j}+u_j')\frac{\partial(\overline{u_i}+u_i')}{\partial x_j}}=\overline{-\frac{1}{\rho}u_i'\frac{\partial(\overline{p}+p')}{\partial x_i}+\nu u_i'\frac{\partial^2(\overline{u_i}+u_i')}{\partial x_j^2}}.$$

各項を展開して

$$\underbrace{\overline{u_i'\frac{\partial\overline{u_i}}{\partial t}}}_{0}+\underbrace{\overline{u_i'\frac{\partial u_i'}{\partial t}}}_{\overline{\frac{\partial}{\partial t}\left(\frac{1}{2}u_i'u_i'\right)}}+\underbrace{\overline{u_i'\overline{u_j}\frac{\partial\overline{u_i}}{\partial x_j}}}_{0}+\underbrace{\overline{u_i'\overline{u_j}\frac{\partial u_i'}{\partial x_j}}}_{\overline{u_j}\overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'u_i'\right)}}+\underbrace{\overline{u_i'u_j'\frac{\partial\overline{u_i}}{\partial x_j}}}_{\overline{u_i'u_j'}\frac{\partial\overline{u_i}}{\partial x_j}}+\overline{u_i'u_j'\frac{\partial u_i'}{\partial x_j}}$$

$$=-\frac{1}{\rho}\underbrace{\overline{u_i'\frac{\partial\overline{p}}{\partial x_i}}}_{0}-\frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_i}}+\underbrace{\overline{\nu u_i'\frac{\partial^2\overline{u_i}}{\partial x_j^2}}}_{0}+\overline{\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}.$$

ここで

$$\overline{u_i'u_j'\frac{\partial u_i'}{\partial x_j}}=\overline{u_j'\frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'u_i'\right)}=\overline{\frac{\partial}{\partial x_j}\left(\frac{1}{2}u_i'u_i'u_j'\right)-\frac{1}{2}u_i'u_i'\underbrace{\frac{\partial u_j'}{\partial x_j}}_{0}}=\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'u_j'}\right),$$

$$-\frac{1}{\rho}\overline{u_i'\frac{\partial p'}{\partial x_i}}=-\frac{1}{\rho}\left[\overline{\frac{\partial}{\partial x_i}(u_i'p')-p'\underbrace{\frac{\partial u_i'}{\partial x_i}}_{0}}\right]=-\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(\overline{u_i'p'}\right)$$

$$\overline{\nu u_i'\frac{\partial^2 u_i'}{\partial x_j^2}}=\nu\left[\overline{\frac{\partial}{\partial x_j}\left(u_i'\frac{\partial u_i'}{\partial x_j}\right)-\left(\frac{\partial u_i'}{\partial x_j}\right)^2}\right]=\nu\frac{\partial}{\partial x_j}\left(\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}$$

の関係を用いて，これは

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right)+\overline{u_j}\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right)$$
$$=-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j'p'}+\frac{1}{2}\overline{u_i'u_i'u_j'}-\nu\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)-\overline{u_i'u_j'}\frac{\partial\overline{u_i}}{\partial x_j}-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2}\tag{5.41}$$

となる．あるいはさらに

$$\frac{\partial}{\partial x_j}\left(\overline{u_i'\frac{\partial u_i'}{\partial x_j}}\right)=\frac{\partial}{\partial x_j}\left(\overline{u_i'\frac{\partial u_i'}{\partial x_j}+u_i'\frac{\partial u_j'}{\partial x_i}}\right)-\frac{\partial}{\partial x_j}\left(\overline{u_i'\frac{\partial u_j'}{\partial x_i}}\right)$$